AP Calculus BC

Polar Equations & Derivatives

1) a) $x^2+y^2=16$

$(r\cos\theta)^2+(r\sin\theta)^2=16$

$r^2\cos^2\theta + r^2\sin^2\theta = 16$

$r^2(\cos^2\theta+\sin^2\theta)=16$

$\boxed{r^2=16}$

b) $4x+3y-1=0$

$4r\cos\theta + 3r\sin\theta - 1 = 0$

$4r\cos\theta + 3r\sin\theta = 1$

$r(4\cos\theta+3\sin\theta)=1$

$\boxed{r = \frac{1}{4\cos\theta+3\sin\theta}}$

c) $y=7$

$r\sin\theta = 7$

$\boxed{r=\frac{7}{\sin\theta}}$

2) a) $r = 3\sec\theta$

$\frac{r}{\sec\theta}=3$

$r\cos\theta = 3$

$\boxed{x = 3}$

b) $4r\cos\theta = r^2$

$4x = x^2+y^2$

$x^2-4x+y^2=0$

$x^2-4x+4+y^2=4$

$\boxed{(x-2)^2+y^2=4}$

c) $\theta=\frac{5\pi}{6}$

$\tan\theta=\frac{y}{x}$

$\tan\frac{5\pi}{6}=\frac{y}{x}$

$\boxed{y=-\frac{\sqrt{3}}{3}x}$

3) a) $r=1-\sin\theta,\ \theta=0$

$x = r\cos\theta$ $\quad y = r\sin\theta$

$x=(1-\sin\theta)\cos\theta$ $\quad y=(1-\sin\theta)\sin\theta$

$\frac{dx}{d\theta}=(1-\sin\theta)(-\sin\theta)+\cos\theta(-\cos\theta)$

$= -\sin\theta+\sin^2\theta-\cos^2\theta$

$\frac{dy}{d\theta}=(1-\sin\theta)\cos\theta+\sin\theta(-\cos\theta)$

$=\cos\theta-2\sin\theta\cos\theta$

$\therefore \frac{dy}{dx}=\frac{\cos\theta-2\sin\theta\cos\theta}{-\sin\theta+\sin^2\theta-\cos^2\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=0}=\frac{1}{-1}=-1$

b) $r = \cos\theta$, $\theta = \frac{\pi}{3}$

$x = \cos^2\theta \qquad y = \sin\theta\cos\theta$

$\frac{dx}{d\theta} = -2\cos\theta\sin\theta \qquad \frac{dy}{d\theta} = -\sin^2\theta + \cos^2\theta$

$\frac{dy}{dx} = \frac{-\sin^2\theta + \cos^2\theta}{-2\cos\theta\sin\theta} \qquad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{-\frac{3}{4} + \frac{1}{4}}{-2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \boxed{\frac{1}{\sqrt{3}}}$

c) $r = 3(1-\cos\theta) \quad \theta = \frac{\pi}{2} \qquad r' = 3\sin\theta$

$x = 3(1-\cos\theta)\cos\theta \qquad y = 3(1-\cos\theta)\sin\theta$

$\frac{dx}{d\theta} = -3(1-\cos\theta)\sin\theta + 3\sin\theta\cos\theta$

$\frac{dy}{d\theta} = 3(1-\cos\theta)\cos\theta + 3\sin^2\theta$

$\frac{dy}{dx} = \frac{3(1-\cos\theta)\cos\theta + 3\sin^2\theta}{-3(1-\cos\theta)\sin\theta + 3\sin\theta\cos\theta}$

$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{3}{-3} = -1$

4) $r = 1 + \sin\theta \qquad r' = \cos\theta$

$x = (1+\sin\theta)\cos\theta$

$\frac{dx}{d\theta} = -(1+\sin\theta)\sin\theta + \cos^2\theta$

$= -\sin\theta - \sin^2\theta + \cos^2\theta = 0$

$= -\sin\theta - \sin^2\theta + 1 - \sin^2\theta = 0$

$-2\sin^2\theta - \sin\theta + 1 = 0$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2} \qquad \sin\theta = -1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \theta = \frac{3\pi}{2}$

Vertical tangents

$\left(\frac{3\sqrt{3}}{4}, \frac{3}{4}\right)\left(-\frac{3\sqrt{3}}{4}, \frac{3}{4}\right)$

$y = (1+\sin\theta)\sin\theta$

$\frac{dy}{d\theta} = (1+\sin\theta)\cos\theta + \cos\theta\sin\theta$

$= \cos\theta + 2\cos\theta\sin\theta = 0$

$\cos\theta(1+2\sin\theta) = 0$

$\cos\theta = 0 \qquad 1 + 2\sin\theta = 0$

$\sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

horizontal tangents

$(0, 2)\left(-\frac{\sqrt{3}}{4}, -\frac{1}{4}\right)\left(\frac{\sqrt{3}}{4}, -\frac{1}{4}\right)$